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ADDED MASS OF A MEMBRANE VIBRATING AT FINITE AMPLITUDE

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The properties of the added mass for a plane membrane suspended horizontally between supports and vibrating naturally in a plane, in air, with a half-sine fundamental mode and finite amplitude are obtained by analysis. Thin aerofoil theory is applied, and the membrane is modelled as a vortex sheet. A ratio of added air-mass to the mass of the membrane itself, α, is defined. It is found that α is equal to 0·68 divided by the mass ratio $m/(\rho l)$, where m is the mass per unit area of the membrane itself, ρ the density of the air flow and l the length of the membrane. It is also found that the added mass is equal to the mass of the air in a rectangle with height 0·68l over the membrane. These results are confirmed by comparison with those obtained from a source-distribution Green's function approach.

1. INTRODUCTION

A MEMBRANE VIBRATING IN AIR moves the ambient air and produces some flow. Consequently, the membrane oscillates with an added mass which is equivalent to the additional mass corresponding to the kinetic energy of the flow. This additional mass is called the *added mass* of the membrane.

An application of the concept is made here to the vibration analysis of a membrane roof subjected to variable wind pressure. For this purpose, the magnitude of the eigen frequency of the membrane roof should be known. In the case where the roof is extremely light, it is possible that the effect of the added air-mass should not be neglected. In the field of design of membrane structures, a knowledge of the effects of added air-mass on the vibration of the membrane is important.

It is well known in the field of fluid dynamics that the added mass of a moving cylinder is equal to the mass of the fluid displaced by the cylinder. Sewall *et al.* (1983) suggest, after comparing the experimental and analytical (using finite element method) results of the vibration frequency of the fundamental mode of a three-sided membrane in air, that the added air-mass, determined using the above-mentioned method for the added mass of a cylinder, is overestimated. Although a number of research papers devoted to the frequency of membranes vibrating in both air and vacuum have been published [e.g., Sygulski (1994)], it seems that research papers related to the magnitude of the added mass of vibrating membranes are very few. Lamb (1921) derived the equations for the added mass on a vibrating circular plate filling an aperture in a plane wall and in contact on one side with water.

The objective of this paper is to investigate analytically the fundamental properties of such added mass of a membrane vibrating in unconfined air. A natural vibration model of the two-dimensional problem, which is a horizontally supported initially plane membrane

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oscillating in an x–y plane with a finite amplitude and a half-sine fundamental mode, is used as the object for this analysis (Figure 1). In order to derive the equation of the air pressure, Δp, working on the vibrating membrane, thin aerofoil theory (Bisplinghoff *et al.* 1955) is applied. Applying the theory, the membrane is modelled as a vortex sheet of γ_m, which has a strength per unit membrane length along the x-axis as illustrated in Figure 1 (Thwaites 1961; Tamada 1966; Kunieda 1975; Minami *et al.* 1994); γ_m is a function of time t and position x along the membrane. Using γ_m distributed along the membrane, an equation for Δp is derived. An equation of the change of kinetic energy of the air-flow is expressed using the equation of Δp and the displacement of the vibrating membrane. Finally, an equation of the added mass, m_f, per unit area of the membrane, corresponding to the kinetic energy of the air is derived. A ratio of m_f to the mass of the membrane itself is defined as the added-mass ratio, α, and this ratio and m_f itself are investigated in this paper.

For verification, the results obtained from the above-mentioned method, based on vortex distribution modelling approach, are compared with those obtained from the usual method, that is a source distribution Green's function approach [e.g., Morse and Ingard (1968)]. The latter approach has been adopted for the analysis of various hydrodynamic and aerodynamic problems. For example, Garrison (1974) and Yeung (1981) calculated the added mass and damping for floating bodies. Sygulski (1994, 1997) analysed the frequency and stability of a membrane oscillating in air considering the added mass.

In the following sections, a detailed description of the analytical method is described. This is followed by sections containing the results of numerical calculations and verification of the method.

2. THE METHOD OF ANALYSIS

2.1. Model of Vibrating Membrane

In a static state, a plane membrane is supported horizontally, with distance l between the supports. The mass per unit area is denoted by m. An x–y coordinate system having the origin at one of the points of support is defined as shown in Figure 1. The x-axis is oriented in the direction of length of the horizontal membrane. The membrane is assumed to be undergoing a natural vibration continuously in the half-sine standing-wave fundamental mode in two-dimensional space, namely the x–y plane. The expression of the displacement, h, at any point x and time t is

$$h(x, t) = a \sin(kx) \cos(\omega t), \quad 0 \le x \le l, \tag{1}$$

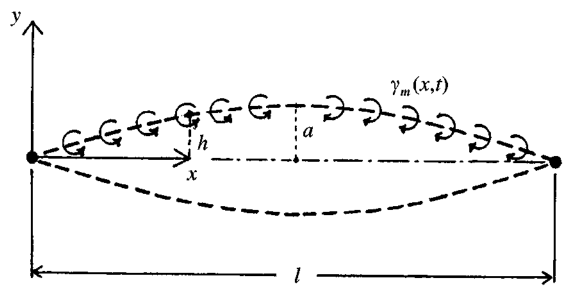

Figure 1. Definition of coordinate system and analytical modelling with a vortex sheet. The vortex strength, $\gamma_m(x, t)$, for natural vibration in the fundamental mode of flat membrane is shown.

where a is the amplitude, $k = \pi/l$ is the wavenumber, $\omega = 2\pi/n_w$ is the circular frequency and n_w is the period.

2.2. Modelling of Membrane as a Vortex Sheet

Consider the pattern of the air-flow formed around the membrane just moving from the upper extreme position toward the lower extreme position. During the first half-period of this motion of finite displacement, the air adjoining the positive-pressure surface of the membrane is exhausted. At the same time, air invades into the vicinity of the negative-pressure surface.

Since the air is assumed to be incompressible, the air-flow from one side around the ends of the membrane compensates for the air invading the other side. This flow pattern is symmetrical with respect to the line of $x = \frac{l}{2}$, which is the centreline perpendicular to the membrane [Figure 2(a)]. The motion during the second half-period is simply reversed [Figure 2(b)].

It is also assumed that the air-flow is inviscid and irrotational. As a result, a velocity potential represents the air-flow in the infinite space outside the membrane.

The velocity components of an air particle in the x and y-directions at any point on the surface outside the boundary layer of the membrane are denoted by u_1 and v_1, respectively, on the lower surface, and by u_2 and v_2, respectively, on the upper surface, as shown in Figure 3. In this figure, A and B are points on the central surface of the membrane, C_1, C_2, D_1, D_2 denote the points on the surface of the respective boundaries. Φ denotes the variable velocity potential of the flow.

The displacement is assumed to be finite, but not large. Thus, applying thin aerofoil theory, the velocity and the velocity potential are treated as variables which take values only on the x-axis. That is to say, these variables are treated as functions of x and time t only.

In the following analysis, subscripts 1 and 2 denote that the corresponding variables are quantities on the lower and upper outside surface of the membrane, respectively. The velocity components u_1 and u_2 are expressed as

$$\left[\frac{\partial \Phi}{\partial x}\right]_1 = u_1, \quad \left[\frac{\partial \Phi}{\partial x}\right]_2 = u_2. \tag{2}$$

The strength of the vortex per unit length along the membrane is γ_m, where positive is defined in this paper to be in the clockwise direction. This vortex strength is a function of x and t so that it can be expressed as $\gamma_m(x, t)$. On the other hand, $\gamma_m\,dx$, the strength of the vortex per infinitesimal length (dx), is a sum of the circulation on the paths ABD_1C_1A and

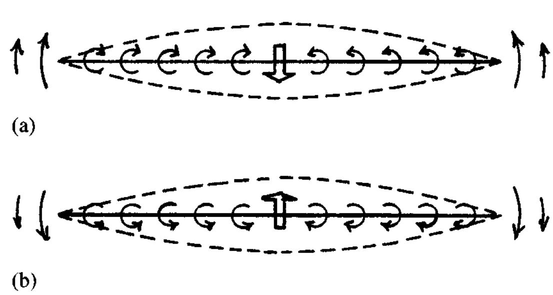

Figure 2. Schematic patterns of flow around the outside ends of the vibrating membrane.

AC_2D_2BA, shown in Figure 3. If it can be assumed that both the membrane and boundary layers are infinitely thin, $\gamma_m\,\mathrm{d}x$ can be expressed as

$$\gamma_m(x, t)\,\mathrm{d}x = (u_2 - u_1)\,\mathrm{d}x, \quad 0 < x < l. \tag{3}$$

The vortex sheet is made up of vortex filaments which extend straight in the direction perpendicular to the x–y plane. The strength of a component vortex is $\gamma_m\,\mathrm{d}x$, and the vortices are distributed in the x-direction along the membrane. The complex velocity potential, f, induced by a vortex filament located at any position $x = \xi$ with the strength $\gamma_m(\xi, \mathrm{t})\,\mathrm{d}\xi$ at any position $z = x + \mathrm{i}y$ (where $\mathrm{i} = \sqrt{-1}$), is expressed as

$$f = \mathrm{i}\kappa \log(z - \xi), \tag{4}$$

where $\kappa = \gamma_m(\xi, t)\,\mathrm{d}\xi/2\pi$. The components of velocity in the x- and y-directions induced by a vortex filament at any point on the membrane are obtained from equation (4) as $\mathscr{Re}[\partial f/\partial z]_{z=x}$ and $\mathscr{Im}[-\partial f/\partial z]_{z=x}$, respectively. The total velocity components in the x-direction (u) and in the y-direction (v) at any point x induced by the vortex filaments distributed along the whole membrane are given as follows:

$$[u]_{y=0} = 0, \quad [v]_{y=0} = \frac{1}{2\pi}\int_0^1 \frac{\gamma_m(\xi, t)}{\xi - x}\,\mathrm{d}\xi, \quad 0 < x < l. \tag{5}$$

To satisfy the left-hand side of the first of equations (5), it is necessary that

$$u_1 + u_2 = 0\cdot \tag{6}$$

The velocity components v_1 and v_2, are shown in Figure 3, are the sums of the moving speed of the inclined membrane and the y-direction velocity component corresponding to the x-direction velocity component u_1 and u_2, respectively. Therefore, v_1 and v_2 can be expressed as

$$v_1 = \frac{\partial h}{\partial t} + u_1\,\frac{\partial h}{\partial x}, \quad v_2 = \frac{\partial h}{\partial t} + u_2\,\frac{\partial h}{\partial x}. \tag{7}$$

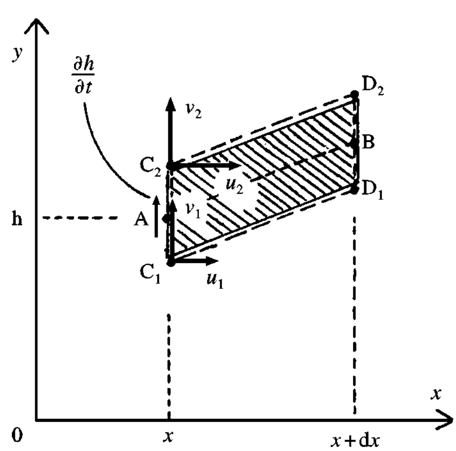

Figure 3. Velocity components and points (which define path for calculation of circulation) on surfaces of the vibrating membrane.

Accordingly, using expressions (7) and (6), the following equation is obtained:

$$\frac{1}{2}(v_1 + v_2) = \frac{\partial h}{\partial t}. \tag{8}$$

Assuming that the membrane is very thin, the expression of equation (8) is approximately

$$[v]_{y=0} = \tfrac{1}{2}(v_1 + v_2). \tag{9}$$

Consequently, using equation (8) and the right-hand side of the second of equations (5), the following singular integral equation results:

$$[v]_{y=0} = \frac{\partial h}{\partial t} = \frac{1}{2\pi}\int_0^l \frac{\gamma_m(\xi, t)}{\xi - x}\,\mathrm{d}\xi, \quad 0 < x < l. \tag{10}$$

In equation (10), the variable $\partial h/\partial t$ is known from equation (1), and γ_m is unknown. The inversion formula [see, for example, Muskhelishvili (1953)] for equation (10) gives the solution as

$$\gamma_m(x, t) = \frac{2}{\pi}\sqrt{\frac{l-x}{x}}\int_0^l \sqrt{\frac{\xi}{l-\xi}}\,\frac{(\partial h/\partial t)}{x-\xi}\,\mathrm{d}\xi + \frac{C't}{\sqrt{x(l-x)}}, \qquad 0 < x < l, \tag{11}$$

where C' is a function of time. Making use of equation (1), equation (11) can be reexpressed as

$$\gamma_m(x, t) = -\frac{2a\omega}{\pi}\sin(\omega t)\left\{\sqrt{\frac{l-x}{x}}\int_0^l \sqrt{\frac{\xi}{l-\xi}}\,\frac{\sin(\mathrm{k}\xi)}{x-\xi}\,\mathrm{d}\xi + \frac{Cl}{\sqrt{x(l-x)}}\right\}, \qquad 0 < x < l, \tag{12}$$

where C is a constant. This constant can be determined by the condition $\gamma_m(\frac{l}{2}, t) = 0$, which holds because the flow pattern is symmetric with respect to the line $x = \frac{l}{2}$. The value of C, determined by numerical calculation, is 0·74142.

Since the flow pattern is symmetric with respect to the line $x = \frac{l}{2}$, γ_m should have values which are of inverse symmetry with respect to a point $x = \frac{l}{2}$. That is, $\gamma_m(x, t) = -\gamma_m(l - x, t)$ along the membrane. This relation means that γ_m satisfies Kelvin's theorem on circulation,

$$\frac{\partial}{\partial t}\int_0^l \gamma_m(x, t)\,\mathrm{d}x = 0. \tag{13}$$

2.3. Pressure Working on the Membrane

The velocity potentials at any point x on both lower and upper surfaces of the membrane can be expressed as

$$[\Phi]_1 = \int_{-\infty}^{x} \mathrm{d}\Phi = \int_{-\infty}^{0} [u]_{y=0}\,\mathrm{d}x + \int_0^x u_1\,\mathrm{d}x \quad \text{and} \quad [\Phi]_2 = \int_{-\infty}^{0} [u]_{y=0}\,\mathrm{d}x + \int_0^x u_2\,\mathrm{d}x, \tag{14}$$

respectively. Substracting $[\Phi]_1$ from $[\Phi]_2$ and then making use of equation (3), the following equation is obtained:

$$[\Phi]_2 - [\Phi]_1 = \int_0^x \gamma_m(x, t)\,\mathrm{d}x, \quad 0 < x < l. \tag{15}$$

Denoting the pressure on the lower surface as p_1 and that on the upper surface as p_2, the generalized Bernoulli's theorem is expressed as follows:

$$\left[\frac{\partial \Phi}{\partial t}\right]_1 + \frac{1}{2}(u_1^2 + v_1^2) + \frac{p_1}{\rho} = \left[\frac{\partial \Phi}{\partial t}\right]_2 + \frac{1}{2}(u_2^2 + v_2^2) + \frac{p_2}{\rho}. \tag{16}$$

Applying equations (3) and (15) to equation (16), the equation for the total pressure (working upward) is as follows:

$$\Delta p = p_1 - p_2 = \rho \frac{\partial h}{\partial t}\frac{\partial h}{\partial x}\gamma_m(x, t) + \rho \frac{\partial}{\partial t}\int_0^x \gamma_m(\xi, t)\, \mathrm{d}\xi, \quad 0 < x < l, \tag{17}$$

where ρ is mass density of the air-flow.

2.4. Definition of Added Mass Ratio α

In this section, the kinetic energy of air and the work to the air by the vibrating membrane, which are both defined as values per unit length perpendicular to the x–y plane, are investigated.

From the energy conservation law, it can be stated that the negative change of kinetic energy of the air, $-\mathrm{d}T_f$, at an instant $\mathrm{d}t$ is equal to the work done on to the whole membrane by the force corresponding to Δp during $\mathrm{d}t$. This relation could be expressed as

$$-\mathrm{d}T_f = \int_0^l \Delta p \, \mathrm{d}s \frac{\partial \mathrm{h}}{\partial t}\mathrm{d}t, \tag{18}$$

where $\mathrm{d}s = \sqrt{1 + (\partial h/\partial x)^2}\,\mathrm{d}x$ at time t.

Denote the added mass per unit area of the membrane as m_f That is to say, the virtual mass of the membrane is $m + m_f$. Since the kinetic energy of the vibrating membrane having a mass equivalent to m_f can be considered to be equal to T_f the following equation could be derived:

$$\mathrm{d}T_f = \frac{\partial}{\partial t}\left\{\frac{1}{2} m_f \int_0^l \left(\frac{\partial h}{\partial t}\right)^2 \mathrm{d}x\right\}\mathrm{d}t = m_f \int_0^l \frac{\partial h}{\partial t}\frac{\partial^2 h}{\partial t^2}\mathrm{d}x\,\mathrm{d}t. \tag{19}$$

Define the added-mass ratio as

$$\alpha = m_f/m. \tag{20}$$

Equating the right-hand sides of equations (18) and (19), and using equation (1), an expression for α is obtained i.e.,

$$\alpha = \frac{2}{mal\omega^2\cos(\omega t)}\int_0^l \Delta p \sin(kx)\sqrt{1 + \left\{\frac{a\pi}{l}\cos(kx)\cos(\omega\tau)\right\}^2}\,\mathrm{d}x. \tag{21}$$

Substituting equations (17) and (12) into (21), α can then be re-expressed as

$$\alpha = \frac{4}{\pi l^2 (m/\rho l)}\int_0^l \left[\frac{\pi a^2}{2l}\sin(2kx)\sin^2(\omega t)\left\{\sqrt{\frac{l-x}{x}}\int_0^l \sqrt{\frac{\xi}{l-\xi}}\frac{\sin(k\xi)}{x-\xi}\mathrm{d}\xi + \frac{Cl}{\sqrt{x(l-x)}}\right\}\right.$$
$$\left. - \int_0^x \left\{\sqrt{\frac{l-\eta}{\eta}}\int_0^l \sqrt{\frac{\xi}{l-\xi}}\frac{\sin(k\xi)}{\eta-\xi}\mathrm{d}\xi + \frac{Cl}{\sqrt{\eta(l-\eta)}}\right\}\mathrm{d}\eta\right]$$
$$\times \sin(kx)\sqrt{1 + \left\{\pi\frac{a}{l}\cos(kx)\cos(\omega t)\right\}^2}\,\mathrm{d}x. \tag{22}$$

Examining equation (22), it can be seen that α depends on two nondimensional parameters: $m/(\rho l)$ and a/l. The parameter $m/(\rho l)$ is called the mass ratio.

In a different way, the quantity m_f can be expressed as a height, H, of the air mounted uniformly on the whole membrane. The mass of the entire layer of this air is $\rho l H$. Dividing H by l, the following nondimensionalized equation involving the height of air H could then be obtained:

$$\frac{H}{l} = \left(\frac{m}{\rho l}\right)\alpha. \tag{23}$$

Hence, the added-mass ratio is expressed as

$$\alpha = \frac{H/l}{(m/\rho l)}. \tag{24}$$

3. RESULTS OF NUMERICAL CALCULATION

The values of the added-mass ratio, α, are calculated by using equation (22) for the case where $m = 1\ \mathrm{kg/m^2}$, $l = 10$ m, and $t = n_w/8$. Gauss's formula is applied for the numerical integrations. The results of α are all the same, having values of 8·34, for values of a/l from 0·01 through 0·1 when n_w equals 1 s. The same α-value is also obtained for values of n_w ranging from 0·1 through 5.0 s when $a/l = 0{\cdot}01$. The values obtained by varying t between 0 and n_w are also obtained, and it is confirmed that α does not depend on the magnitude of t in equation (22). By studying these results of α, it is also confirmed that α does not depend on the period n_w or a/l.

Further calculations of α, varying m and l, confirm that α depends uniquely on the mass ratio $m/(\rho l)$. The relationship between α and $m/(\rho l)$ is shown in Figure 4. In the figure, α is 0·83 in two cases (those being $l = 10$ m and $m = 10\ \mathrm{kg/m^2}$, and $l = 0{\cdot}5$ m and $m = 0{\cdot}5\ \mathrm{kg/m^2}$), while the mass ratio, $m/(\rho l)$, has a value of 0·82.

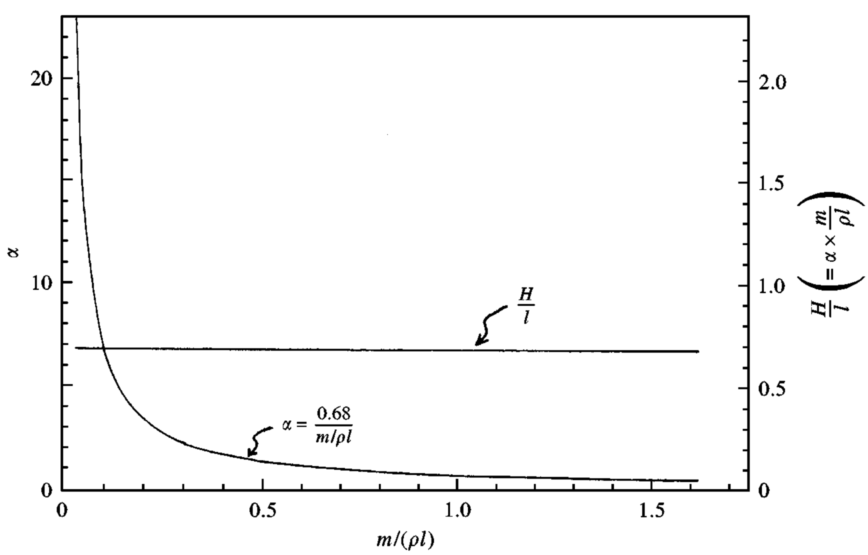

Figure 4. Relationships obtained by numerical calculations between mass ratio, $m/\rho l$, and added-mass ratio, α, and relationships between mass ratio and H/l.

The results of H/l obtained from equation (23) are also presented in Figure 4. This ratio has a value 0·68 for any length and mass of the membrane.

Results of numerical calculation for the distribution of γ_m and Δp along the vibrating membrane are presented in Figure 5. The results of γ_m and Δp shown in Figure 5 are for the case with $l = 50$ cm, $n_w = 0{\cdot}2$ s and $a = 1$ cm. It can be seen that these results satisfy the

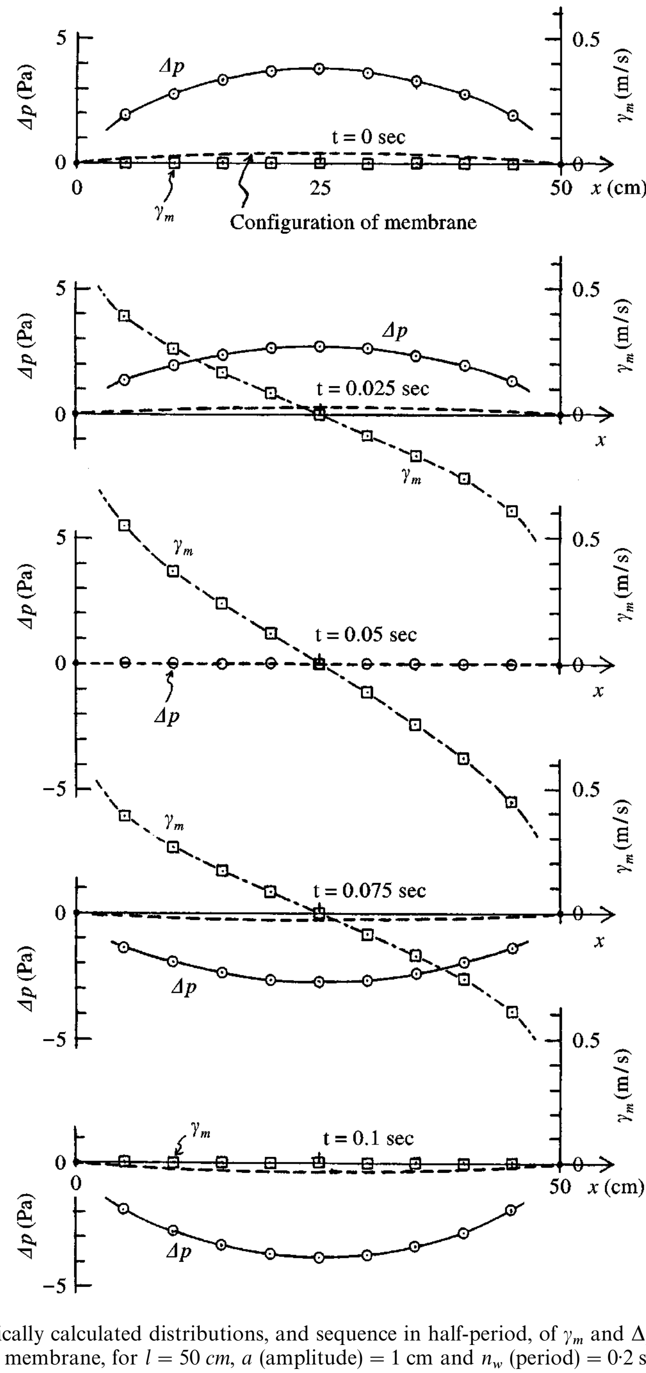

Figure 5. Numerically calculated distributions, and sequence in half-period, of γ_m and Δp along the vibrating membrane, for $l = 50$ *cm*, a (amplitude) = 1 cm and n_w (period) = 0·2 s.

following requirements: γ_m should satisfy the Kelvin's theorem in equation (13), and its distribution should have an inversely symmetric configuration. γ_m should be infinite at the ends of the membrane, because the air-flow has been assumed as inviscid. Δp should be zero at the ends of the membrane, and it is natural to suppose that the distribution of Δp is nearly uniform near the centre of the membrane. Furthermore, the sequence of change of the work done by Δp should be similar to that of the inertia force acting on a vibrating mass particle suspended from a simple spring.

4. VERIFICATION OF THE ANALYTICAL METHOD

In the foregoing sections, the analytical method for the two-dimensional problem of the added mass of a membrane vibrating in unconfined air has been presented, and the added mass has been calculated. The analytical method makes use of a vortex distribution model. This method is verified in this section by comparing the results of the previous section and those obtained from a source-distribution Green's function approach which is the usual analytical method used.

The source distribution modelling and the definition of orthogonal coordinates system are shown in Figure 6. As shown in this figure, a membrane which is initially flat in the x–z plane and vibrates with half-sine modes in both the x- and the z-directions with an amplitude, a, is considered. This membrane model has length, l, in the x-direction which is the same as that used in the two-dimensional model analysed in the previous section, and it has length βl in the z-direction, which is considered to be sufficiently long. Therefore, it can be considered that the flow condition around the membrane in the x–y plane at $z = 0{\cdot}5\ \beta l$ is approximately equivalent to that around the two-dimensional model in the previous section.

In this section, the following expression for the displacement of vibrating membrane is used in place of expression (1), i.e.

$$h(x, z, t) = a \sin(kx)\, sin\left(\frac{\pi}{\beta l} z\right) \cos(\omega t), \quad 0 \leq x \leq l, \quad 0 \leq z \leq \beta l. \tag{25}$$

A half-space ($y \geqq 0$) denoted by V is considered, with boundaries defined as in Figure 6. The boundaries are those at the infinity and on the x–z plane. The boundary on the x–z plane is denoted by S and is divided into five domains which are denoted by $S_{al}(-\infty < x < 0,\ 0 \leqq z \leqq \beta l)$, $S_{a2}(l < x < \infty,\ 0 \leqq z \leqq \beta l)$, $S_{b1}(-\infty < z < 0)$, $S_{b2}(\beta l < z < \infty)$ and $S_m(0 \leq x \leq l, 0 \leqq z \leqq \beta l)$.

The velocity potential, Φ', at any point, $P(x, y, z)$, in V can be expressed by the well-known equation

$$\Phi'(P) = -\frac{1}{2\pi} \iint_S \frac{\partial \Phi'}{\partial n} \frac{1}{r(P, Q)}\, \mathrm{d}s, \tag{26}$$

where $S = S_{a1} + S_{a2} + S_{b1} + S_{b2} + S_m$ and r is the distance between P and a point, Q, on the boundary, S. For the sake of the simplification of treatment, it is assumed that the upper surface of the vibrating membrane is equal to S_m and that S_{b1} and S_{b2} are the surfaces of the rigid buffles. Accordingly, the velocity potential in equation (26) can be written as follows:

$$\Phi'(P) = -\frac{1}{2\pi}\left[\int_0^{\beta 1}\left\{\int_{-\infty}^{0} \frac{\partial \Phi'}{\partial y}\frac{1}{r(P, Q)}\,\mathrm{d}\xi + \int_0^{\infty}\frac{\partial \Phi'}{\partial y}\frac{1}{r(P, Q)}\,\mathrm{d}\xi + \int_0^{l}\frac{\partial h}{\partial t}\frac{1}{r(P, Q)}\,\mathrm{d}\xi\right\}\mathrm{d}\zeta\right], \tag{27}$$

where ξ and ζ are integral variables corresponding to x and z, respectively.

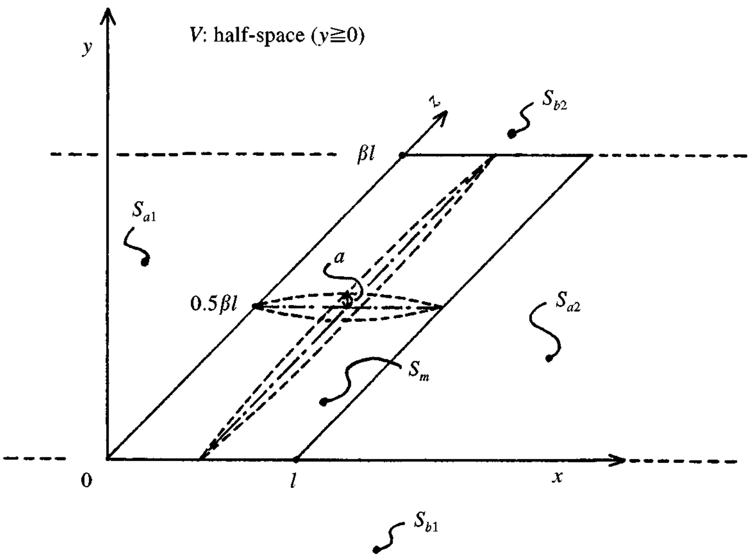

Figure 6. Definition of coordinate system, and source distribution modelling with a half-space and boundaries.

Referring to the function for the change of the velocity which decreases inversely with the distance from the centre of a vortex filament, the function for the distribution of the velocity component normal to S_{a1} and S_{a2} are assumed, respectively, as follows:

$$\left[\frac{\partial \Phi'}{\partial y}\right]_{S_{a1}} = \begin{cases} \left(\dfrac{\lambda_1}{x}\right)\sin\left(\dfrac{\pi}{\beta l} z\right) & (-\varepsilon_1 l \leq x \leq -\varepsilon_2 l), \\ 0 & (-\infty < x < -\varepsilon_1 l, \quad -\varepsilon_2 l < x < 0) \end{cases} \tag{28a}$$

and

$$\left[\frac{\partial \Phi'}{\partial y}\right]_{S_{a2}} = \begin{cases} \left(\dfrac{\lambda_2}{x-l}\right)\sin\left(\dfrac{\pi}{\beta l} z\right) & (l+\varepsilon_2 l \leq x \leq l+\varepsilon_1 l), \\ 0 & (l < x < l+\varepsilon_2 l, \quad l+\varepsilon_l l < x < \infty), \end{cases} \tag{28b}$$

where ε_1 and ε_2 are constants given for the limitation of the domain where the distribution of the normal velocity exists. By this limitation of the domain, λ_1 and λ_2 can be determined by equating the mass of the flow passing through S_{a1} and S_{a2} per unit time to that displaced by the membrane vibration. It will be appropriate to take ε_1 to be several multiples of l, and ε_2 to be very small, that is $\varepsilon_2 \ll 1$. Thus, the determined λ_1 and λ_2 could be expressed as

$$\lambda_1 = \frac{a\omega l}{\pi \log(\varepsilon_2/\varepsilon_1)} \sin(\omega t) = -\lambda_2. \tag{29}$$

Consequently, by substituting equations (28) and (25) into equation (27), which is the equation for the velocity potential, a flow pattern that is approximately the same as that obtained from the previous vortex distribution model approach can be obtained.

Because of the symmetry of the flow pattern, the kinetic energy of the air in the total space around the membrane is twice of that in the half-space, V. Also the total kinetic energy is equal to the kinetic energy of the vibrating membrane with the mass m_f. Referring to equation (19), this relation can be expressed as follows:

$$2\left[-\frac{\rho}{2}\left\{\int_0^{\beta l}\int_{-\varepsilon_1 l}^{-\varepsilon_2 l}\Phi'\frac{\partial \Phi'}{\partial y}\,d\xi\,d\zeta+\int_0^{\beta l}\int_{l+\varepsilon_2 l}^{l+\varepsilon_1 l}\Phi'\frac{\partial \Phi'}{\partial y}\,d\xi\,d\zeta+\int_0^{\beta l}\int_0^{l}\Phi'\frac{\partial h}{\partial t}\,d\xi\,d\zeta\right\}\right]$$
$$=\int_0^{\beta l}\int_0^{l}\frac{1}{2}m_f\left(\frac{\partial h}{\partial t}\right)^2 dx\,dz. \tag{30}$$

Consequently, by substituting equations (27) and (25) into (30), the expression for the added-mass ratio is obtained as follows:

$$\alpha=\frac{4}{\pi\beta l^3(m/\rho l)}$$
$$\times\left[\frac{-2l}{\pi\log(\varepsilon_2/\varepsilon_1)}\int_0^{\beta l}\sin\left(\frac{\pi}{\beta l}z\right)\int_{-\varepsilon_1 l}^{-\varepsilon_2 l}\frac{1}{x}\left\{\int_0^{\beta l}\sin\left(\frac{\pi}{\beta l}\zeta\right)\right.\right.$$
$$\left.\times\left\{\frac{-l}{\pi\log(\varepsilon_2/\varepsilon_1)}(I_{a1}-I_{a2})+I_m\right\}d\zeta\right\}dx\,dz$$
$$+\int_0^{\beta l}\sin\left(\frac{\pi}{\beta l}z\right)\int_0^{l}\sin(kx)\left\{\int_0^{\beta l}\sin\left(\frac{\pi}{\beta l}\zeta\right)\right.$$
$$\left.\left.\times\left\{\frac{-l}{\pi\log(\varepsilon_2/\varepsilon_1)}(I_{a1}-I_{a2})+I_m\right\}d\zeta\right\}dx\,dz\right], \tag{31}$$

where

$$I_{a1}=\int_{-\varepsilon_1 l}^{-\varepsilon_2 l}\frac{1}{\xi\sqrt{(x-\xi)^2+(z-\zeta)^2}}\,d\xi,\quad I_{a2}=\int_{l+\varepsilon_2 l}^{l+\varepsilon_1 l}\frac{1}{(\xi-1)\sqrt{(x-\xi)^2+(z-\zeta)^2}}\,d\zeta,$$
$$I_m=\int_0^{l}\frac{\sin(k\xi)}{\sqrt{(x-\xi)^2+(z-\zeta)^2}}\,d\xi,$$

This equation shows that α does not depend on l and the period of vibration. However, since this method is established approximately based on the assumption that the membrane surface is considered to be on S_m in x–z plane, the amplitude does not appear in this equation.

The method presented in the previous section can be verified by comparison between the values of H/l. Figure 7 shows the results of H/l calculated from equation (23) using α obtained from equation (31) taking $\varepsilon_2 = 0{\cdot}001$. Two relationships, which are between β and H/l and between ε_1 and H/l, are shown in this figure. According to these relationships, it can be seen that $\beta = 3$ is an adequate value for the objectives of this section; furthermore, although H/l does not show the tendency of convergence with ε_1, it will be possible here to adopt the value of $\varepsilon_1 = 3$ since the actual y-direction velocity of the point at a distance of three times of l from the ends of the membrane will be considered to be sufficiently small. As a consequence it can be said that the result of H/l in the case of the membrane having a sufficient length in the z-direction is approximately 0·94.

The result of H/l being obtained in the foregoing section, that is 0·68, can be compared with the above result, 0·94. Although the difference between these results is not small, it can be confirmed, taking into consideration the existence of some approximations taken into the method in this section, that the method presented in the previous section is appropriate.

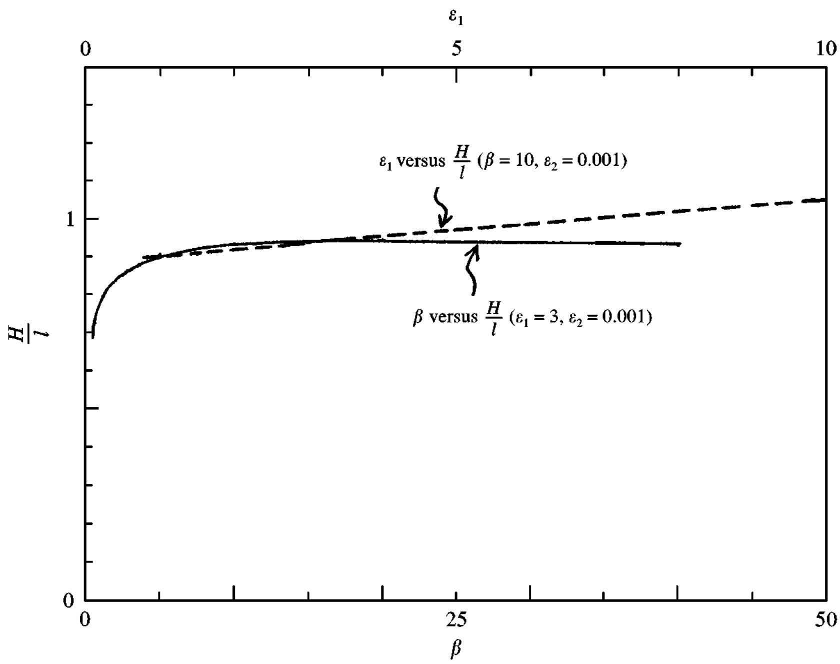

Figure 7. Variation of H/l calculated with varying β or ε_1.

5. CONCLUSIONS

The added air mass of a horizontally supported, initially plane membrane undergoing finite amplitude natural vibration in the half-sine fundamental mode is investigated by two-dimensional analysis. Thin aerofoil theory is applied, where the amplitude of the displacement is up to 10% of the length of the membrane. From results of numerical analysis, the following conclusions are obtained:

(i) the added-mass ratio α uniquely depends on mass ratio $m/(\rho l)$;
(ii) α does not depend on the period and the amplitude of the oscillation;
(iii) the height of an equivalent air layer corresponding to the added mass distributed uniformly over the membrane is equal to 68% of the membrane length. Accordingly, α is given by the expression $\alpha = 0{\cdot}68/[m/(\rho l)]$, from equation (24).

The above-mentioned method is verified by comparison with the results obtained approximately from these usual approach, involving a source distribution and using Green's function.

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APPENDIX: NOMENCLATURE

a	amplitude of membrane vibrating in half-sine mode
C	a constant; see equation (11)
C'	a function of t; see equation (11)
$\mathrm{d}s$	infinitesimal length of membrane vibrating
f	complex velocity potential of a vortex filament
H	height of air equivalent to added mass mounted uniformly on whole membrane
h	displacement of membrane: see equations (1) and (25)
I_{a1}	see equation (31)
I_{a2}	see equation (31)
I_m	see equation (31)
i	$(-1)^{0\cdot 5}$
k	wavenumber, l/k
l	initial length of membrane in the x-direction
m	mass of membrane per unit area
m_f	mass of membrane per unit area equivalent to added mass
n_w	period of vibration
P	any point in half-space (V)
Δp	pressure difference between lower and upper surfaces of membrane, equation (17)
p_1	pressure on lower surface of membrane
p_2	pressure on upper surface of membrane
Q	point on the x–z plane
r	distance between points, [P] and [Q]
S	boundary that is x–z plane, equation (26)
S_{a1}	a domain in S; see Figure 6
S_{b1}	a domain in S; see Figure 6
S_{a2}	a domain in S; see Figure 6
S_{b2}	a domain in S; see Figure 6
S_m	a domain in S; see Figure 6
T_f	kinetic energy of air
t	time
u	velocity component of air-flow in the x-direction
u_1	velocity component of air-flow in the x-direction on the lower surface of the membrane
u_2	velocity component of air-flow in the x-direction on the upper surface of the membrane
v	velocity component of air-flow in the y-direction
v_1	velocity component of air-flow in the y-direction on the lower surface of the membrane
v_2	velocity component of air-flow in the y-direction on the upper surface of the membrane

x	a coordinate; see Figures 1 and 6
y	a coordinate; see Figures 1 and 6
z	a coordinate; see Figure 6
α	added-mass ratio
β	multiple to l; see Figure 6
γ_m	strength of vortex per unit length along the membrane
ε_1	multiplier to l; see equation (28)
ε_2	multiplier to l; see equation (28)
ζ	integral variable corresponding to z.
η	integral variable corresponding to x
λ_1	constant; see equation (28)
λ_2	constant; see equation (28)
ξ	integral variable corresponding to x
ρ	mass density of air-flow
Φ	variable velocity potential; see equation (2) or (14)
Φ'	variable velocity potential; see equation (26)
ω	circular frequency